Two-Phase, Annular, Laminar Flow of Simple Fluids Through Cylindrical Tubes

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The flow of two phases through pipe lines is an important problem, both since this is a common industrial situation and since pumping costs for very viscous fluids can often be reduced by introducing a less viscous second phase. The easiest two-phase tube flow problem to consider is axisymmetric, annular laminar flow. Unfortunately the practical realization of this situation depends both upon the preferential wetting of the pipe wall by one of the phases (hopefully the less viscous phase in order that the pressure drop be reduced) and (except in the less common case of vertical flow) upon the absence of an appreciable density difference between the liquids. In most situations probably some stratification takes place, which results in a far different pressure drop vs. volume rate of flow relationship; in annular flow a small amount of the phase which preferentially wets the wall wets the entire wall, but in stratified flow a small amount of a second phase wets only a small portion of the wall.

The case where both phases are Newtonian has received some attention (see reference *1* both for a summary of previous work and for a numerical solution of the stratified flow problem), but nothing appears to have been done about the problem where both phases are non-Newtonian. This paper treats the simplest case of two-phase flow (annular, two-phase, laminar flow) for a very general class of fluids, the simple fluid of Noll (*2* to *6*).

For an incompressible, simple fluid the extra stress $\boldsymbol{\tau} = \mathbf{t} + p\mathbf{I}$ at a particular material point at time t depends upon the entire past history of the right Cauchy-Green strain tensor at the material point relative to the configuration at time t (*4*). The simple fluid has several conceptual advantages; it includes all previously proposed constitutive equations for fluids (equations which represent the behavior of the material by expressing stress in the material as a function of deformation), and it yields inviscid behavior and Newtonian behavior as zeroth and first approximations respectively to a simple fluid which has been nearly at rest for all past time (*4*). It has the disadvantage that, without further assumptions, so far solutions have been found only for two classes of flow, the viscometric flows (*3* to *6*) and a class of extension problems (*7*, *8*). The problem discussed below is included within the class of viscometric flows.

ANALYSIS

The problem is to be viewed in cylindrical coordinates, where the positive z axis is taken in the direction of flow and coincides with the axis of the cylinder. For $0 < r < \overline{R}$, one has phase 1; for $\overline{R} < r < R$, phase 2. The velocity feld is time independent, the phases are incompressible, and the behavior of each phase is assumed to be described by Noll's theory of simple fluids (*2* to *8*).

One begins by assuming that there is only one nonzero component of velocity v_z, which is in turn a function of r. Under this restriction Coleman and Noll (*3 to 6*) have shown that there are only four nonzero components of the extra stress tensor $\boldsymbol{\tau}$ (τ_{rr}, $\tau_{\theta\theta}$, τ_{zz}, τ_{rz}) and that the behavior of the material may be represented in terms of three material functions:

$$\tau = \tau(\kappa) = -\tau(-\kappa) = t_{rz} \quad (1)$$

$$\sigma_1 = \sigma_1(\kappa) = \sigma_1(-\kappa) = \tau_{rr} - \tau_{\theta\theta} = t_{rr} - t_{\theta\theta} \quad (2)$$

$$\sigma_2 = \sigma_2(\kappa) = \sigma_2(-\kappa) = \tau_{zz} - \tau_{\theta\theta} = t_{zz} - t_{\theta\theta} \quad (3)$$

$$\kappa = dv_z/dr \quad (4)$$

If one further assumes that the external body force per unit mass $\mathbf{f}$ may be represented by a potential

$$\mathbf{f} = -\nabla\varphi \quad (5)$$

the three components of the equation of motion reduce to

$$0 = -\frac{\partial P}{\partial r} + \frac{d\tau_{rr}}{dr} + \frac{(\tau_{rr} - \tau_{\theta\theta})}{r} \quad (6)$$

$$0 = -\frac{\partial P}{\partial \theta} \quad (7)$$

$$0 = -\frac{\partial P}{\partial z} + \frac{1}{r}\frac{d}{dr}(r\,t_{rz}) \quad (8)$$

where

$$P = p + \rho\varphi \quad (9)$$

Since by Equations (6) and (8) $\partial P/\partial r$ and $\partial P/\partial z$ are functions only of r, one concludes that

$$\text{phase 1:} \quad -\frac{\partial P}{\partial z} = a_{(1)} = \text{constant} \quad (10)$$

$$\text{phase 2:} \quad -\frac{\partial P}{\partial z} = a_{(2)} = \text{constant} \quad (11)$$

Using Equations (10) and (11) one may integrate Equation (8) to obtain

$$\text{phase 1:} \quad r\,t_{rz(1)} = -a_{(1)} r^2/2 + b_{(1)} \quad (12)$$

$$\text{phase 2:} \quad r\,t_{rz(2)} = -a_{(2)} r^2/2 + b_{(2)} \quad (13)$$

One has introduced here $b_{(1)}$ and $b_{(2)}$ as constants of integration. Equation (12) is simplified somewhat by noting that $(r\,t_{rz})$ must go to zero at $r = 0$, and

$$b_{(1)} = 0 \quad (14)$$

From Equations (6) and (10)

$$P_{(1)} - P_{(1)}\Big|_{\substack{r=\overline{R}\\ r=0}} = \int_0^z \frac{\partial P}{\partial z}dz + \int_{\overline{R}}^r \frac{\partial P}{\partial r}dr \quad (15)$$

$$= -a_{(1)}z + \int_{\overline{R}}^r \frac{d\tau_{rr(1)}}{dr}dr + \int_{\overline{R}}^r \frac{\sigma_{1(1)}(\kappa)}{r}dr \quad (16)$$

or after some rearrangement

$$t_{rr(1)} = t_{rr(1)}\Big|_{\substack{r=\overline{R}\\ z=0}} + a_{(1)}z + \rho_{(1)}\varphi - \rho_{(1)}\varphi\Big|_{\substack{r=\overline{R}\\ z=0}} + \int_r^{\overline{R}} \frac{\sigma_{1(1)}(\kappa)}{r}dr \quad (17)$$

In the same way, from Equations (6) and (11)

$$t_{rr(2)} = t_{rr(2)} \Big|_{r=0}^{r=\bar{R}} + a_{(2)} z + \rho_{(2)} \varphi - \rho_{(2)} \varphi \Big|_{z=0}^{r=\bar{R}} + \int_r^{\bar{R}} \frac{\sigma_{1(2)}(\kappa)}{r} dr \quad (18)$$

Conservation of mass and of momentum at an interface between two phases has been discussed at length by Scriven (*9*) [see also (*10* to *12*)]. If mass transfer is neglected across the interface, and if it is assumed that there are no gradients in surface tension in the interface, the requirements of conservation of momentum in the interface reduce for this case to (see *10*, p. 243)

$$t_{rz(1)}|_{r=\bar{R}} = t_{rz(2)}|_{r=\bar{R}} \quad (19)$$

$$t_{rr(1)}|_{r=\bar{R}} - t_{rr(2)}|_{r=\bar{R}} = -\sigma/\bar{R} \quad (20)$$

Equations (12) through (14) and (19) yield

$$b_{(2)} = (\bar{R})^2 (a_{(2)} - a_{(1)})/2 \quad (21)$$

Equations (17), (18), and (20) give

$$[a_{(1)} - a_{(2)}] z + [\rho_{(1)} - \rho_{(2)}] [\varphi \Big|_{}^{r=\bar{R}} - \varphi \Big|_{z=0}^{r=\bar{R}}] = 0 \quad (22)$$

This equation is satisfied in only two cases. In the first the densities of the two phases are equal, and Equations (21) and (22) yield

$$a_{(1)} = a_{(2)}, \; b_{(2)} = 0 \quad (22a)$$

This situation will not be discussed further, because the analysis is parallel to that given below. In the second case gravity acts along the axis of the cylinder (in the negative z direction here), and from Equations (21) and (22) one obtains

$$a_{(1)} + \rho_{(1)} g = a_{(2)} + \rho_{(2)} g \quad (23)$$

$$b_{(2)} = \bar{R}^2 [\rho_{(1)} - \rho_{(2)}] g/2 \quad (24)$$

Going back to Equations (1), (12), (13), (14), (23), and (24) one finds

$$\tau_{(1)}(\kappa) = t_{rz(1)} = -a_{(1)} r/2 \qquad \text{for } r < \bar{R} \quad (25)$$

$$\tau_{(2)}(\kappa) = t_{rz(2)} = -a_{(2)} r/2 + b_{(2)}/r \quad \text{for } r > \bar{R} \quad (26)$$

Under fairly general conditions (*4*, sec. 3) $\tau(\kappa)$ will have a single-valued inverse, and

$$\kappa = \frac{dv_z}{dr} = -\tau_{(1)}^{-1}[a_{(1)} r/2] \qquad \text{for } r < \bar{R} \quad (27)$$

$$\kappa = -\tau_{(2)}^{-1}[a_{(2)} r/2 - b_{(2)}/r] \qquad \text{for } r > \bar{R} \quad (28)$$

The volume flow rate Q of material in two-phase annular flow through a tube may be written as

$$Q = 2\pi \int_{\bar{R}}^{R} v_z r\, dr + 2\pi \int_0^{\bar{R}} v_z r\, dr \quad (29)$$

or after an integration by parts

$$Q = \pi \int_{\bar{R}}^{R} r^2 (-dv_z/dr) dr + \pi \int_0^{\bar{R}} r^2 (-dv_z/dr) dr \quad (30)$$

By Equations (27) and (28) this last becomes

$$\frac{Q\, a_{(2)}^3}{8\pi} = \int_{a_{(2)}\bar{R}/2}^{a_{(2)}R/2} \xi^2 \tau_{(2)}^{-1}[\xi - b_{(2)} a_{(2)}/(2\xi)]\, d\xi + \left[\frac{a_{(2)}}{a_{(1)}}\right]^3 \int_0^{a_{(1)}\bar{R}/2} \xi^2 \tau_{(1)}^{-1}[\xi]\, d\xi \quad (31)$$

The quantities $a_{(1)}$ and $b_{(2)}$ may be eliminated from this equation by means of (23) and (24) to leave a relation between Q and $a_{(2)}$.

The argument which Coleman and Noll (*3, 4*) used to obtain a physical interpretation of a in flow between two flat plates and in Poiseuille flow may be extended readily to this case. The z component of force applied to the fluid lying between two planes $z = z_I$ and $z = z_{II}$ is given by

$$F_z = -\int_A t_{zz} \Big|_{z=z_I} dA + \int_A t_{zz} \Big|_{z=z_{II}} dA - \int_{z_I}^{z_{II}} \int_A \frac{\partial \varphi}{\partial z} \rho\, dA\, dz \quad (32)$$

$$= \int_A [\{t_{zz} - \rho\varphi\}_{z=z_{II}} - \{t_{zz} - \rho\varphi\}_{z=z_I}]\, dA \quad (33)$$

Equations (2), (3), and (17) give

$$r < R: t_{zz(1)} = \sigma_{2(1)}(\kappa) - \sigma_{1(1)}(\kappa) + t_{rr(1)} \Big|_{z=0}^{r=\bar{R}} + a_{(1)} z + \rho_{(1)} \varphi - \rho_{(1)} \varphi \Big|_{z=0}^{r=\bar{R}} + \int_r^{\bar{R}} \frac{\sigma_{1(1)}(\kappa)}{r} dr \quad (34)$$

Evaluating the integrals in Equation (33) by Equation (34) and the similar expression for $r > \bar{R}$ one arrives at

$$a_{(1)} = \frac{F_{z(1)}}{\pi \bar{R}^2 (z_{II} - z_I)}, \quad a_{(2)} = \frac{F_{z(2)}}{\pi (R^2 - \bar{R}^2)(z_{II} - z_I)} \quad (35)$$

This shows us that a_i should be interpreted as the driving force in the direction of flow per unit area per unit length of conduit for phase i.

The results given above simplify correctly to those for the movement of a single phase through a tube (3) as either $\bar{R} \rightarrow 0$ or $\bar{R} \rightarrow R$.

SUMMARY

Assume that the first material function $\tau = \tau(\kappa)$ defined by Equation (1) has been measured for two particular materials in a previous experiment, perhaps in some other geometry (*3 to 6*). This information could be used to evaluate the integrals in Equation (31) and, after elimination of $a_{(1)}$ and $b_{(2)}$ by Equations (23) and (24), to obtain a specific relation between the volume flow rate, Q, and the driving force $a_{(2)}$ [defined by either Equation (11) or Equation (35)] for simple fluids of Noll in two-phase, annular, laminar flow through a tube when gravity acts along the axis of the tube. Somewhat simpler results may be derived in the same manner for the case where the densities of the two phases are equal. [See Equation (22a).]

An interesting point brought out by this analysis is that no more experimental information is required to use Equation (31) for fluids of very general behavior than for fluids of a very restricted class of behaviors, such as a restricted class of incompressible Stokesian fluids (*13*, page 231)*

$$\tau = \eta\, \mathbf{d} \quad (36)$$

$$\eta = \eta(\text{II}), \quad \text{II} = \mathbf{d}:\mathbf{d} \quad (37)$$

$$\mathbf{d} = [\nabla \mathbf{v} + (\nabla \mathbf{v})^t]/2 \quad (38)$$

which includes the common empirical models for fluid behavior such as the power model (*14*, page 102).

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* By $(\nabla \mathbf{v})_t$ is meant the transpose of $\nabla \mathbf{v}$.

NOTATION

$a, a_{(1)}, a_{(2)}$ = defined by Equations (23), (10), and (11)
$b_{(1)}, b_{(2)}$ = constants of integration in Equations (12) and (13)
$\mathbf{d}$ = rate-of-deformation tensor
$\mathbf{f}$ = body force vector per unit mass
F_z = physical component of force in z direction driving the fluid through the tube
$F_{z(i)}$ = F_z for phase i
p = pressure, defined for an incompressible fluid as $p = -1/3$ trace $(\mathbf{t})$
P = defined by Equation (9)
Q = voume rate of flow
r, z = cylindrical coordinates
R = radius of tube
$\overline{R}$ = radius of interface between the two phases
$\mathbf{v}$ = velocity vector
v_z = physical component of the velocity vector in z direction in cylindrical coordinates
$\mathbf{t}$ = stress, a second-order tensor
$t_{rr}, t_{\theta\theta}, t_{zz}, t_{rz}$ = physical components of $\mathbf{t}$ in cylindrical coordinates

Greek Letters

θ = cylindrical coordinate
κ = defined by Equation (4)
ρ = density
σ = surface tension
$\sigma_1 = \sigma_1(\kappa)$, $\sigma_2 = \sigma_2(\kappa)$, $\tau = \tau(\kappa)$ = material functions defined by Equations (1), (2), and (3)
τ = extra stress, a second-order tensor defined by $\tau = \mathbf{t} + p\mathbf{I}$ where $\mathbf{I}$ is the unit tensor
$\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}, \tau_{rz}$ = physical components of τ in cylindical coordinates
τ^{-1} = single-valued inverse of τ which is assumed to exist (*4*, sec. 3)
φ = body force potential defined by Equation (5)

LITERATURE CITED

1. Gemmell, A. R., and Norman Epstein, *Can. J. Chem. Eng.*, **40**, 215 (1962).
2. Noll, Walter, *Arch. Rational Mech. Anal.*, **2**, 197 (1958).
3. Coleman, B. D., and Walter Noll, *ibid.*, **3**, 289 (1959).
4. ———, *Annals, N. Y. Acad. Sci.*, **89**, 672 (1961).
5. Coleman, B. D., *Arch. Rational Mech. Anal.*, **9**, 273 (1962).
6. Noll, Walter, *ibid.*, **11**, 97 (1962).
7. Coleman, B. D., and Walter Noll, *Phys. of Fluids*, **5**, 840 (1962).
8. Slattery, J. C., *Phys. of Fluids*, to be published.
9. Scriven, L. E., *Chem. Eng. Sci.*, **12**, 98 (1960).
10. Aris, Rutherford, "Vectors, Tensors, and the Basic Equations of Fluid Mechanics," Prentice-Hall, Englewood Cliffs, New Jersey (1962).
11. Slattery, J. C., *Chem. Eng. Sci.*, **19**, 379 (1964).
12. *Ibid.*, **19**, 453 (1964).
13. Serrin, James, "Handbuch der Physik," S. Flügge, ed., Vol. VIII/1, Springer-Verlag, Berlin (1960).
14. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," Wiley, New York (1960).

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The Mechanics of Vertical Gas-Liquid Fluidized System I: Countercurrent Flow

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Fluidized systems are encountered frequently in the chemical and petroleum industries in cases where it is necessary to transfer heat or matter from one phase to another immiscible or partially miscible phase or to carry out a chemical reaction on the surface of a catalyst. Because of the complexity of a detailed mathematical analysis, the designer of an industrial fluidized system must always rely on experimental data obtained from a small scale model of the system. A research program has been under way for a number of years at Princeton to develop a generalized theory of fluidized systems. It attempts to point out the similarities which exist between different fluidized systems so that data from one type can be used in the design of others. The program has been concerned with systems in which both phases move in a vertical direction, since this is the most common situation found in industrial equipment.

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